CFD applied to Turbomachinery

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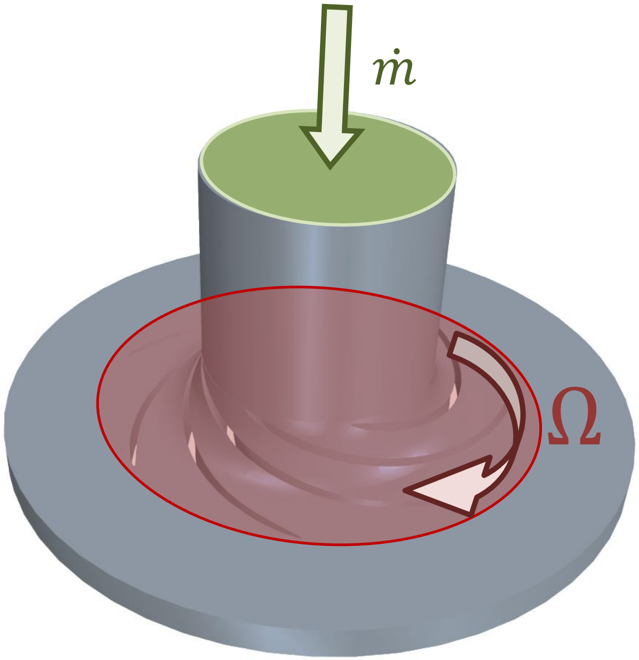


March 9, 2021, International Master of Turbulence

Motivation and Challenges

Rotating parts and fixed parts

High Reynolds numbers

Compressible effects

Complicated geometries

Large mesh size (up to several million cells)

CFD for industrial applications: (U)RANS and a few (Z)DNES

CFD for scientific research: (U)RANS and (Z)DNES for complex geometries
LES and DNS in simplified geometries

Pavesi et al Energies **2016**, 9, 534; doi:10.3390/en9070534

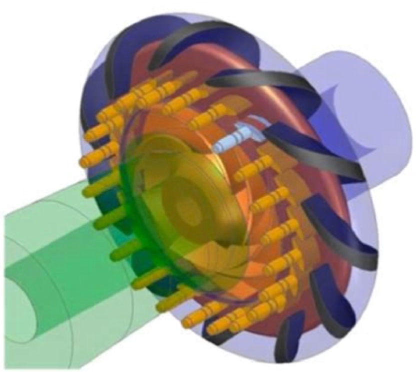


Figure 1. 3D scheme of the tested configuration.

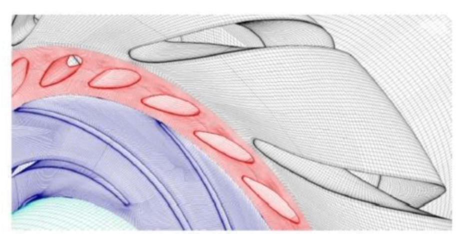


Figure 3. Detail of the coarse mesh of the numerical model: inlet duct, impeller, wicket gates and return channel.

Mesh size: 3.96 million cells

Turbulence model: DES

Time step: rotation of 1°

Reversible pump-turbine

Simulations in Turbomachinery

Rotating parts and fixed parts

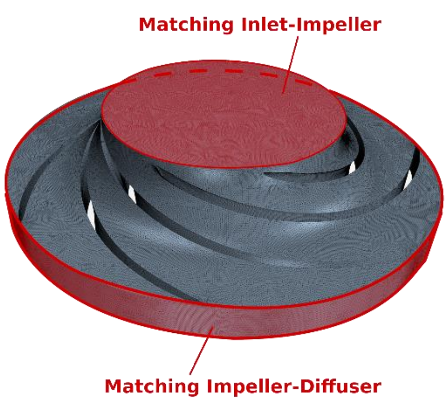
- Matching steady and rotating reference frames
- Momentum equation for incompressible Newtonian flows

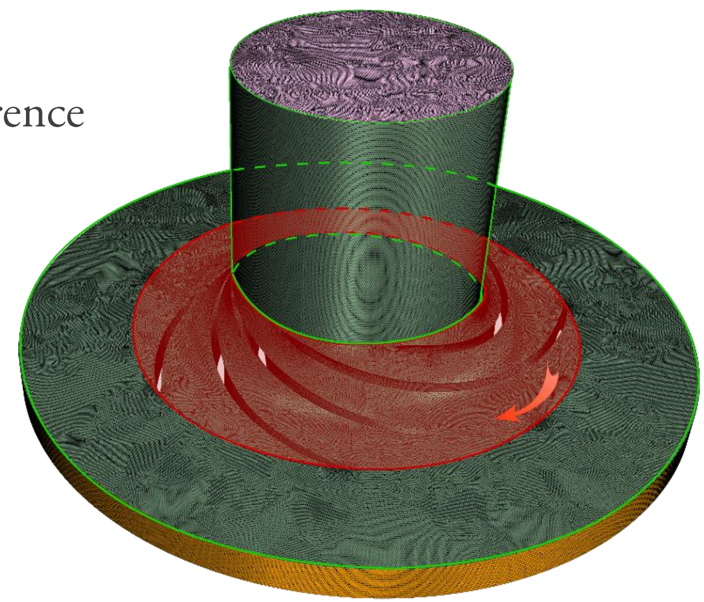
fixed frame: $\rho(\partial_t \boldsymbol{U} + \boldsymbol{U} \cdot \nabla \boldsymbol{U}) = -\nabla p + \mu \nabla^2 \boldsymbol{U}$

rotating frame: $\rho(\partial_t \boldsymbol{U}_r + \boldsymbol{U}_r \cdot \nabla \boldsymbol{U}_r + \underbrace{2\boldsymbol{\omega} \times \boldsymbol{U}_r}_{\text{Coriolis force}} + \underbrace{\boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{r}}_{\text{centrifugal force}}) = -\nabla p + \mu \nabla^2 \boldsymbol{U}_r$

Fixed parts solved in a fixed frame of reference

Rotating parts solved in a rotating frame of reference





Simulations in Turbomachinery

Spatial periodicity near rotor-stator stage

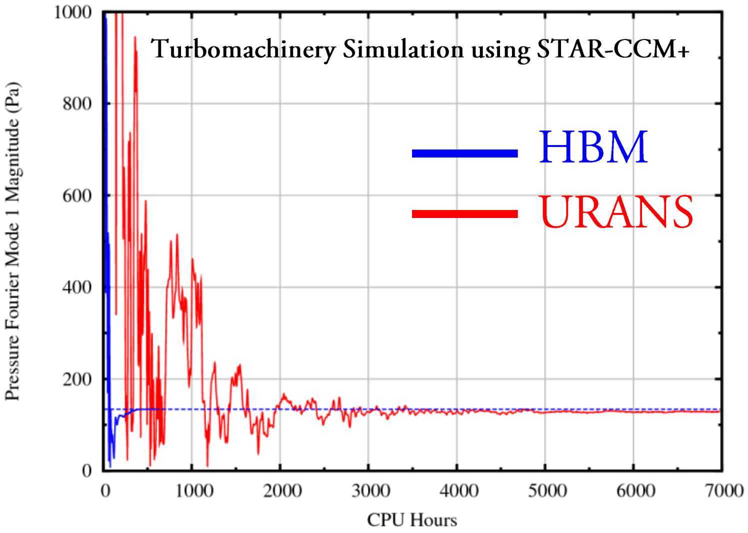
It is possible to use the spatial periodicity of the problem to decrease the size of the problem, e.g. simulating only one blade passage is simulated with periodical boundary conditions.

Reynolds-averaged simulations: steady (RANS) or unsteady (URANS)

Many turbomachinery simulations are performed as stationary simulations. Transient simulations are required when the global flow is strongly influenced by unsteadiness, e.g. rotor-stator interaction effects, large unsteady separations

Harmonic Balance (HB) method

Modern techniques take advantage of spatial periodic domains to induce fast convergence of unsteady simulations (factor 10). Further advantage of the HB is the memory requirement. For the URANS all the blades must be simulated, while for HB only one blade passage is considered.



He, L. (1992). Method of simulating unsteady turbomachinery flows with multiple perturbations. *AIAA journal*, *30*(11), 2730-2735.

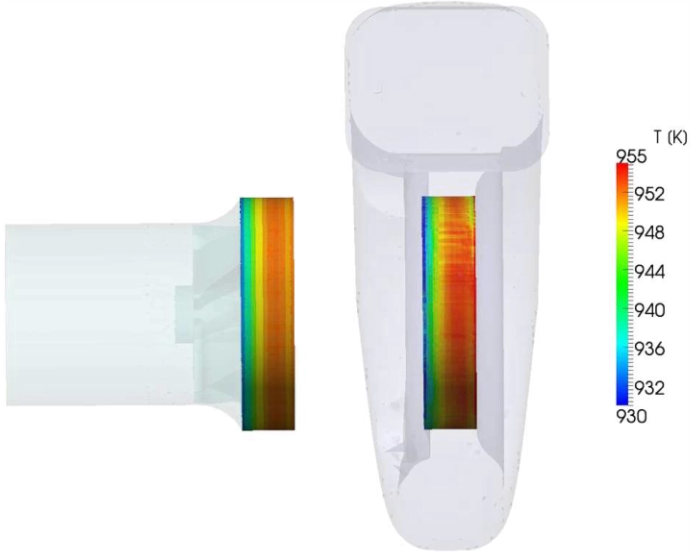
Classic Approximations in Steady Simulations

Frozen rotor simulations

In a frozen rotor simulation the rotating and the stationary parts have a fixed relative position. With a frozen-rotor simulation rotating wakes, secondary flows, leading edge pressure increases will always stay in the same relative positions. This makes a frozen rotor simulation very dependent on how the rotors and the stators are positioned. Frozen rotor simulations are mainly performed to obtain a good starting flow-field before doing a transient sliding mesh simulation.

Steady mixing-plane simulations

Since the mixing-plane method was first introduced, it has become the industry standard type of rotor-stator simulations. A mixing-plane simulation is steady and only requires one rotor blade and one stator blade per stage. Between the rotating blade passage and the steady vane passage the flow properties are circumferentially averaged in a so-called mixing plane interface. This will of course remove all transient rotor-stator interactions, but it still gives representative results.



Denton, J. D. (2010, October). Some limitations of turbomachinery CFD. In Turbo Expo: Power for Land, Sea, and Air (Vol. 44021, pp. 735-745).

Classic Approximations in Unsteady Simulations

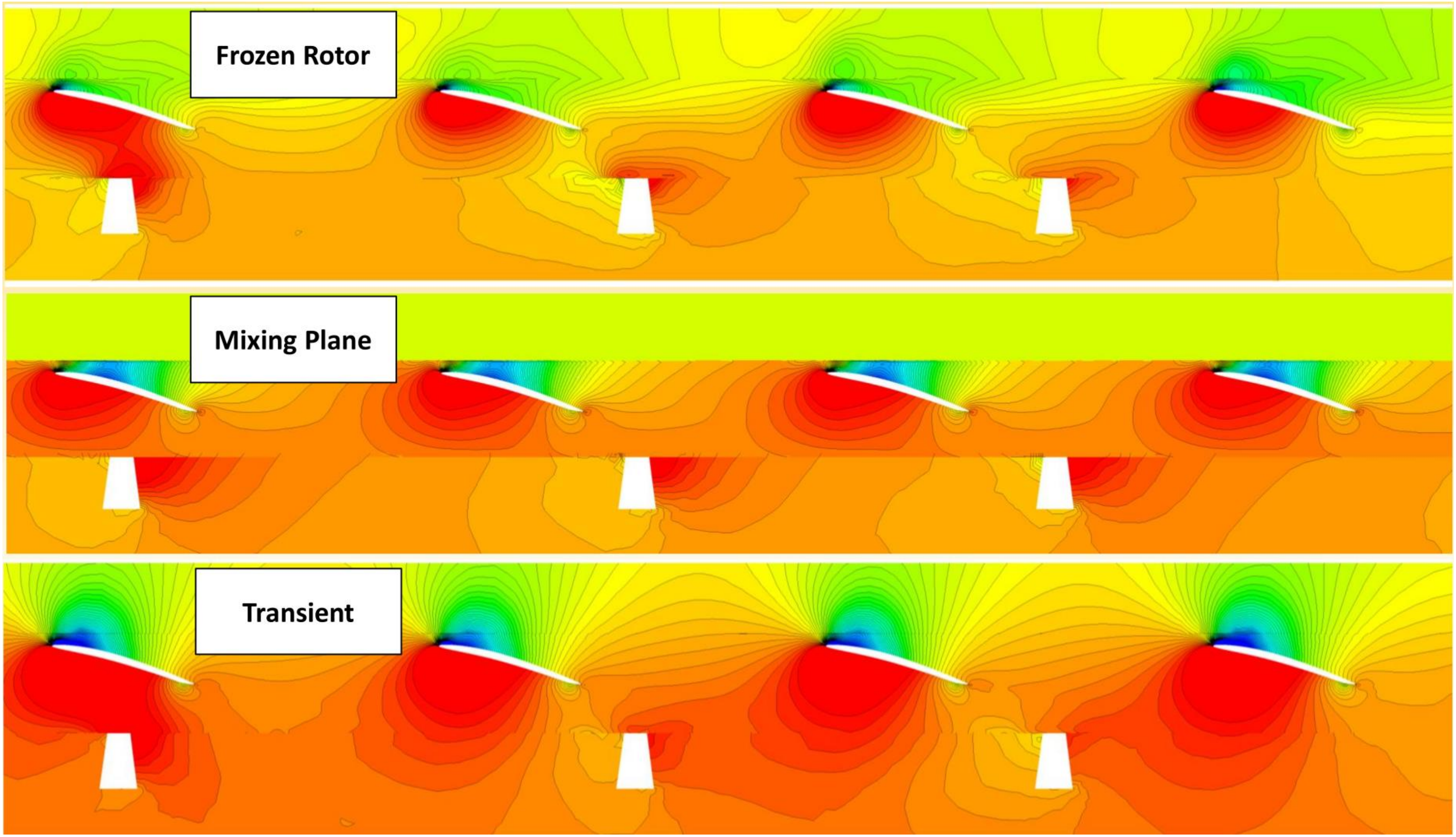
Unsteady sliding-mesh stator-rotor simulations

This is the most complete type of stator-rotor simulation. In most engines the number of stators and rotors do not have a common denominator (to avoid instabilities caused by resonance between different rings). Hence, to make a full unsteady sliding-mesh computation it is necessary to have a mesh which includes the full wheel with all the stators vanes and all rotor blades. This is often not possible, instead it is necessary to reduce the number of vanes and blades by finding a denominator that is almost common and then scale the geometry slightly circumferentially.

Example:

Real engine: 36 stator vanes, 41 rotor blades
Approximated engine: 41 stator vanes, 41 rotor blades, making it possible to simulate only 1 stator vane and 1 rotor blade
Scaling of stator: All stator vanes are scaled by 36/41 = 0.8780 circumferentially

Comparison between Classic Approximations



Turbomachinery Simulation using STAR-CCM+

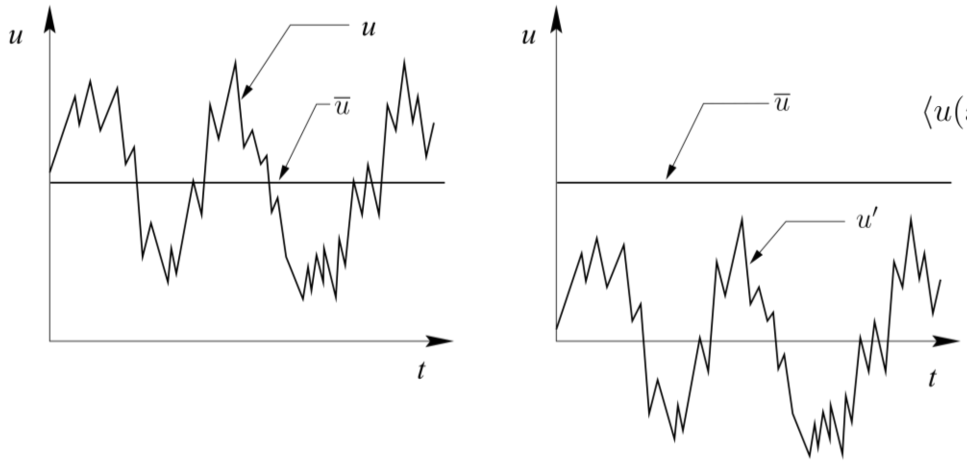
Difference between (U)RANS and LES

Reynolds decomposition (U)RANS: $u(\boldsymbol{x},t) = \overline{u}(\boldsymbol{x}) + u'(\boldsymbol{x},t)$

where $\overline{u}(\boldsymbol{x}) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T u(\boldsymbol{x},t)\,dt$

ergodicity

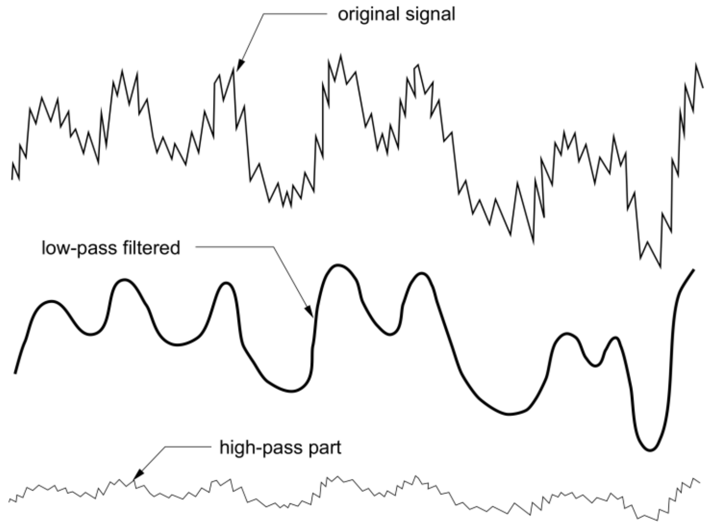
$$\langle u(\boldsymbol{x},t)\rangle \equiv \frac{1}{N}\sum_{i=1}^{N} u^{(i)}(\boldsymbol{x},t)$$



U. Frisch. Turbulence, the Legacy of A. N. Kolmogorov, Cambridge University Press, Cambridge, 1995.

Large Eddy decomposition LES: $u(\boldsymbol{x},t) = \widetilde{u}(\boldsymbol{x},t) + u'(\boldsymbol{x},t)$

where $\widetilde{u}(\boldsymbol{x},t) \equiv \frac{1}{V_\Omega}\int_\Omega u(\boldsymbol{\xi},t)G(\boldsymbol{x}|\boldsymbol{\xi})\,d\boldsymbol{\xi}$



Reynolds Averaged for Incompressible Flows

Continuity equation: $\nabla \cdot \boldsymbol{U} = 0$

Navier-Stokes equation: $\boldsymbol{U}_t + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U}$

Reynolds decomposition (U)RANS: $u(\boldsymbol{x}, t) = \overline{u}(\boldsymbol{x}) + u'(\boldsymbol{x}, t)$ where $\overline{u}(\boldsymbol{x}) \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T u(\boldsymbol{x}, t)\, dt$

Applying the Reynolds decomposition to the continuity equation

$$\nabla \cdot \boldsymbol{U} = \nabla \cdot (\overline{\boldsymbol{u}} + \boldsymbol{u'}) = \nabla \cdot \overline{\boldsymbol{u}} + \nabla \cdot \boldsymbol{u'} = 0$$

Then averaging this equation results in

$$\nabla \cdot \overline{\overline{\boldsymbol{u}}} + \nabla \cdot \overline{\boldsymbol{u'}} = 0$$

Considering the property of the average operator

$$\overline{\overline{u}} = \overline{u}\,, \qquad \text{and} \qquad \overline{u'} = 0 \longrightarrow \nabla \cdot \overline{\boldsymbol{u}} = 0$$

Subtracting it from the Reynolds decomposed continuity equation, it yields

$$\nabla \cdot \boldsymbol{u'} = 0$$

Reynolds Averaged for Incompressible Flows

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Reynolds decomposition (U)RANS: $u(\boldsymbol{x}, t) = \overline{u}(\boldsymbol{x}) + u'(\boldsymbol{x}, t)$ where $\overline{u}(\boldsymbol{x}) \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T u(\boldsymbol{x}, t)\, dt$

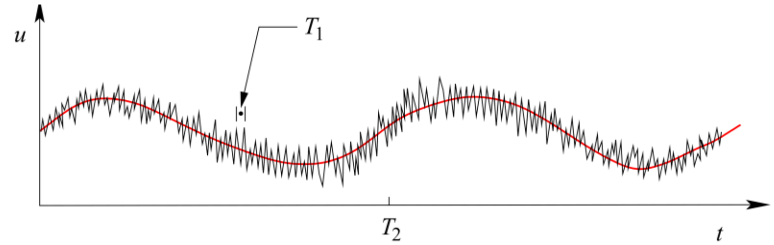
Applying the Reynolds decomposition to the Navier-Stokes equation

$$(\overline{\boldsymbol{u}} + \boldsymbol{u}')_t + (\overline{\boldsymbol{u}} + \boldsymbol{u}') \cdot \nabla (\overline{\boldsymbol{u}} + \boldsymbol{u}') = -\nabla (\overline{p} + p') + \nu \Delta (\overline{\boldsymbol{u}} + \boldsymbol{u}')$$

for steady RANS $\partial_t \overline{\boldsymbol{u}} = \boldsymbol{0}$

hypothesis of URANS:

mean and turbulence time scales are different



For RANS: Expanding the convective term (we consider RANS below)

$$\boldsymbol{u}'_t + \overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}} + \overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \overline{\boldsymbol{u}} + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' = -\nabla (\overline{p} + p') + \nu \Delta (\overline{\boldsymbol{u}} + \boldsymbol{u}')$$

Then averaging this equation results in

$$\overline{\boldsymbol{u}'_t} + \overline{\overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}}} + \overline{\overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}'} + \overline{\boldsymbol{u}' \cdot \nabla \overline{\boldsymbol{u}}} + \overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} = -\nabla \overline{(\overline{p} + p')} + \nu \Delta \overline{(\overline{\boldsymbol{u}} + \boldsymbol{u}')}$$

Considering that fluctuations are zero-average ($\overline{u'} = 0$) the first term is null

$$\overline{\overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}}} + \overline{\overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}'} + \overline{\boldsymbol{u}' \cdot \nabla \overline{\boldsymbol{u}}} + \overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} = -\nabla \overline{p} + \nu \Delta \overline{\boldsymbol{u}}$$

Reynolds Averaged for Incompressible Flows

Continuity equation: $\nabla \cdot \boldsymbol{U} = 0$

Navier-Stokes equation: $\boldsymbol{U}_t + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U}$

Reynolds decomposition (U)RANS: $u(\boldsymbol{x},t) = \overline{u}(\boldsymbol{x}) + u'(\boldsymbol{x},t)$ where $\overline{u}(\boldsymbol{x}) \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T u(\boldsymbol{x},t)\, dt$

Applying the Reynolds decomposition to the Navier-Stokes equation

$$\overline{\overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}}} + \overline{\overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}'} + \overline{\boldsymbol{u}' \cdot \nabla \overline{\boldsymbol{u}}} + \overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} = -\nabla \overline{p} + \nu \Delta \overline{\boldsymbol{u}}$$

Applying the definition of the time average, we find out that

$$\begin{aligned}
\overline{\overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}'} &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}'\, dt \\
&= \overline{\boldsymbol{u}} \cdot \left[\lim_{T\to\infty} \frac{1}{T} \int_0^T \nabla \boldsymbol{u}'\, dt \right] \\
&= \overline{\boldsymbol{u}} \cdot \nabla \left[\lim_{T\to\infty} \frac{1}{T} \int_0^T \boldsymbol{u}'\, dt \right] \\
&= \overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}'} \\
&= 0
\end{aligned}$$

(same applies to the other fluctuation-mean mixed term)

Reynolds Averaged for Incompressible Flows

Continuity equation: $\nabla \cdot \boldsymbol{U} = 0$

Navier-Stokes equation: $\boldsymbol{U}_t + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U}$

Reynolds decomposition (U)RANS: $u(\boldsymbol{x}, t) = \overline{u}(\boldsymbol{x}) + u'(\boldsymbol{x}, t)$ where $\overline{u}(\boldsymbol{x}) \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T u(\boldsymbol{x}, t)\, dt$

Applying the Reynolds decomposition to the Navier-Stokes equation

$$\overline{\overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}}} + \overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} = -\nabla \overline{p} + \nu \Delta \overline{\boldsymbol{u}}$$

$$\overline{\overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}}} = \overline{\nabla \cdot \overline{\boldsymbol{u}}^2} \qquad \overline{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} = \overline{\nabla \cdot \boldsymbol{u}'^2}$$

Applying the definition of the time average, we find out that

$$\overline{\nabla \cdot \overline{\boldsymbol{u}}^2} = \nabla \cdot \overline{\overline{\boldsymbol{u}}^2} = \nabla \cdot \overline{\boldsymbol{u}}^2 \qquad \text{and} \qquad \overline{\nabla \cdot \boldsymbol{u}'^2} = \nabla \cdot \overline{\boldsymbol{u}'^2}$$

Summarizing, we arrive to the following Reynolds averaged Navier-Stokes (RANS) equation

$$\nabla \cdot \overline{\boldsymbol{u}}^2 + \nabla \cdot \overline{\boldsymbol{u}'^2} = -\nabla \overline{p} + \nu \Delta \overline{\boldsymbol{u}}$$

Reynolds Averaged for Incompressible Flows

Continuity equation: $\nabla \cdot \boldsymbol{U} = 0$

Navier-Stokes equation: $\boldsymbol{U}_t + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U}$

Reynolds decomposition (U)RANS: $u(\boldsymbol{x},t) = \overline{u}(\boldsymbol{x}) + u'(\boldsymbol{x},t)$ where $\overline{u}(\boldsymbol{x}) \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T u(\boldsymbol{x},t)\, dt$

Applying the Reynolds decomposition to the continuity and Navier-Stokes equations

Continuity equations: $\nabla \cdot \overline{\boldsymbol{u}} = 0$ and $\nabla \cdot \boldsymbol{u}' = 0$

RANS equation: $\nabla \cdot \overline{\boldsymbol{u}}^2 = -\nabla \overline{p} + \nu \Delta \overline{\boldsymbol{u}} - \boldsymbol{R}(\boldsymbol{u}', \boldsymbol{u}')$

or

URANS equation: $\overline{\boldsymbol{u}}_t + \nabla \cdot \overline{\boldsymbol{u}}^2 = -\nabla \overline{p} + \nu \Delta \overline{\boldsymbol{u}} - \boldsymbol{R}(\boldsymbol{u}', \boldsymbol{u}')$

where the last term of the Reynolds averaged NS is the divergence of the Reynolds stress tensor

$$\boldsymbol{R}(\boldsymbol{u}', \boldsymbol{u}') = \nabla \cdot \begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{pmatrix}$$

Turbulence closure problem: modeling the Reynolds stress tensor

Recap of the Einstein Summation

Cartesian tensor notation: $\boldsymbol{x} = (x_1, x_2, x_3)^T$ instead of $\boldsymbol{x} = (x, y, z)^T$

$\boldsymbol{U} = (u_1, u_2, u_3)^T$ instead of $\boldsymbol{U} = (u, v, w)^T$

Einstein summation: summation of repeated indices

$$\nabla \cdot \boldsymbol{U} = \frac{\partial u_i}{\partial x_i} \equiv \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \quad \left(= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

$$S_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \longrightarrow \boldsymbol{S}\,\boldsymbol{S} = \sum_{k=1}^{3} S_{ik} S_{kj} \qquad \forall\ i, j = 1, 2, 3$$

Incompressible continuity and Navier-Stokes equations in Cartesian tensor notation

$$\frac{\partial u_i}{\partial x_i} = 0\,,$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}\,, \qquad i = 1, 2, 3$$

Reynolds Averaged for Incompressible Flows

Continuity equation: $$\frac{\partial u_i}{\partial x_i} = 0\,,$$

Navier-Stokes equation: $$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}\,, \qquad i = 1, 2, 3$$

Applying the Reynolds decomposition to the continuity and Navier-Stokes equations

Continuity equations: $$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial u'_i}{\partial x_i} = 0$$

RANS equation: $$\frac{\partial}{\partial x_j} \overline{u}_i \overline{u}_j = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial R_{ij}}{\partial x_j}\,, \quad i = 1, 2, 3$$

or

URANS equation: $$\overline{u}_{i,t} + \frac{\partial}{\partial x_j} \overline{u}_i \overline{u}_j = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} - \frac{\partial R_{ij}}{\partial x_j}\,, \quad i = 1, 2, 3$$

where the last term of the Reynolds averaged momentum includes the Reynolds stress tensor

$$\boldsymbol{R} = \begin{pmatrix} \overline{u'_1 u'_1} & \overline{u'_1 u'_2} & \overline{u'_1 u'_3} \\ \overline{u'_1 u'_2} & \overline{u'_2 u'_2} & \overline{u'_2 u'_3} \\ \overline{u'_1 u'_3} & \overline{u'_2 u'_3} & \overline{u'_3 u'_3} \end{pmatrix}$$

Turbulence closure problem: modeling the Reynolds stress tensor

Reynolds Averaged for Compressible Flows

Change of notation to discriminate compressible flow equations:

$$U_i = \bar{U}_i + u_i \qquad P = \bar{P} + p$$

Continuity and momentum equation for compressible flows:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\rho U_i)_{,i} &= 0 \\ \frac{\partial \rho U_i}{\partial t} + (\rho U_i U_j)_{,j} &= -P_{,i} + \left[\mu \left(U_{i,j} + U_{j,i} - \frac{2}{3}\delta_{ij} U_{k,k}\right)\right]_{,j} \end{aligned}$$

where the Kronecker delta is defined as follows: $\delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{otherwise} \end{cases} \qquad i, j = 1, 2, 3$

Applying the Reynolds decomposition to the continuity and Navier-Stokes equations:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\rho \bar{U}_i)_{,i} &= 0 \\ \frac{\partial \rho \bar{U}_i}{\partial t} + \left(\rho \bar{U}_i \bar{U}_j\right)_{,j} &= -\bar{P}_{,i} + \left[\mu(\bar{U}_{i,j} + \bar{U}_{j,i}) - \rho \overline{u_i u_j}\right]_{,j} \end{aligned}$$

Turbulence closure problem: modeling the Reynolds stress tensor terms $R_{ij} = \rho \overline{u_i u_j}$

Classic Modeling Assumptions

Boussinesq hypothesis:

Small-scale turbulent stress is linearly proportional to the mean large-scale strain rates

$$\overline{u'v'} \sim \frac{1}{2}\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)$$

This is somehow the analogous of Newton's law of viscosity that relates strain and stress tensors. However, there is no physical reason why such assumption must hold for linking turbulent fluctuations and mean large-scale flow.

Turbulent eddy viscosity ν_T:

Assuming that the Boussinesq hypothesis holds, we can define the turbulent eddy viscosity as a coefficient of proportionality between mean large-scales and fluctuations

$$-\overline{u'v'} = \nu_T\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)$$

The turbulent eddy viscosity is nothing, but a definition derived from the analogy with laminar flows (kinematic viscosity ν) and does not have basis in physics. Indeed, the eddy viscosity changes with each flow, hence it is not a physical property of the fluid as the kinematic viscosity ν.

Limits of the Boussinesq Hypothesis

Boussinesq hypothesis:

Small-scale turbulent stress is linearly proportional to the mean large-scale strain rates

$$\overline{u'v'} \sim \frac{1}{2}\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)$$

This is somehow the analogous of Newton's law of viscosity that relates strain and stress tensors. However, there is no physical reason why such assumption must hold for linking turbulent fluctuations and mean large-scale flow.

Several flows have proven inadequate the Boussinesq hypothesis:

- Flows with sudden changes in mean strain rate
- Flows over curved surfaces
- Flows in ducts
- Flows having secondary fluid motions, including boundary-layer separation
- Flows of rotating and/or stratified fluids
- Flows with a strong three-dimensional component

Algebraic Models

Prandtl mixing-length theory:

Based on an analogy between turbulent eddies and molecules or atoms of a gas, it makes use of kinetic theory to determine the length and velocity scales needed to derive the turbulent eddy viscosity. It is an analogous to the first-principles derivation of molecular viscosity (kinetic theory) but applied to fluctuations and large-scale mean flows.

Based on dimensional analysis, for 2D shear flows the mixing length scales like:

$$v_{mix} \sim \ell_{mix} \left| \frac{d\overline{u}}{dy} \right| \qquad \nu_T \sim \ell_{mix} v_{mix} \qquad \nu_T = \ell_{mix}^2 \left| \frac{d\overline{u}}{dy} \right| \qquad \left| -\overline{u'v'} \right| = \left| \ell_{mix} \frac{d\overline{u}}{dy} \right|^2$$

Generalizing: $\ell_{mix} = C_1 \delta(x)$

Flow Type	Far Wake	Mixing Layer	Plane Jet	Round Jet
C_1	0.180	0.071	0.098	0.080

Using the Boussinesq hypothesis: $\frac{\partial \overline{u}^2}{\partial x} + \frac{\partial \overline{u}\,\overline{v}}{\partial y} = -\frac{\partial \overline{p}}{\partial x} + \frac{\partial}{\partial x}\left((\nu + \nu_T) \frac{\partial \overline{u}}{\partial x} \right) + \frac{\partial}{\partial y}\left((\nu + \nu_T) \frac{\partial \overline{u}}{\partial y} \right)$

As its predictive performances are quite poor, but the rationale of the mixing-length theory is clear and solid up to a certain extend, the mixing length theory is used to analyze simple flows. Other estimates have been proposed for the mixing length based on the Karman constant or the Van Driest damping function.

One-Equation Models

Spalart-Allmaras model:

A transport equation is solved for determining the turbulent kinetic energy and algebraic arguments are used to obtained the turbulent length scale.

$$\nu_t = \tilde{\nu} f_{v1}, \quad f_{v1} = \frac{\chi^3}{\chi^3 + C_{v1}^3}, \quad \chi := \frac{\tilde{\nu}}{\nu}$$

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = C_{b1}[1 - f_{t2}]\tilde{S}\tilde{\nu} + \frac{1}{\sigma}\{\nabla \cdot [(\nu + \tilde{\nu})\nabla\tilde{\nu}] + C_{b2}|\nabla\tilde{\nu}|^2\} - \left[C_{w1} f_w - \frac{C_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + f_{t1}\Delta U^2$$

$$\tilde{S} \equiv S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}, \quad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \qquad S = \sqrt{2\Omega_{ij}\Omega_{ij}}$$

$$f_w = g\left[\frac{1 + C_{w3}^6}{g^6 + C_{w3}^6}\right]^{1/6}, \quad g = r + C_{w2}(r^6 - r), \quad r \equiv \frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2} \qquad f_{t1} = C_{t1} g_t \exp\left(-C_{t2}\frac{\omega_t^2}{\Delta U^2}[d^2 + g_t^2 d_t^2]\right) \qquad f_{t2} = C_{t3}\exp(-C_{t4}\chi^2)$$

$$\begin{aligned} \sigma &= 2/3 & \kappa &= 0.41 & C_{w3} &= 2 & C_{t2} &= 2 \\ C_{b1} &= 0.1355 & C_{w1} &= C_{b1}/\kappa^2 + (1 + C_{b2})/\sigma & C_{v1} &= 7.1 & C_{t3} &= 1.1 \\ C_{b2} &= 0.622 & C_{w2} &= 0.3 & C_{t1} &= 1 & C_{t4} &= 2 \end{aligned}$$

- Generally simple, economical, robust on good meshes
- Valid in the near-wall region
- Good predictions for attached flow
- Typical applications: airfoil and axial turbomachinery performance (aerospace industry)

Two-Equations Models

Standard $k-\varepsilon$ model:

Two transport equations are solved for determining the turbulent kinetic energy k and the dissipation rate $\varepsilon = k^{3/2}/l$. Even though exact equations can be derived, too many unknown coefficients would make the model unpractical for computational reasons, hence the exact production and dissipation terms are simplified based on scaling considerations and on the Boussinesq hypothesis. The remaining constants of the simplified model are assumed universal.

$$(\rho \bar{U}_j k)_{,j} = \left[\left(\mu + \frac{\mu_t}{\sigma_k}\right) k_{,j}\right]_{,j} + P_k - \rho\varepsilon \qquad \text{where} \quad \frac{1}{2}\overline{\rho u_j u_i u_i} = -\frac{\mu_t}{\sigma_k} k_{,j}$$

$$(\rho \bar{U}_j \varepsilon)_{,j} = \left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right) \varepsilon_{,j}\right]_{,j} + \frac{\varepsilon}{k}\left(c_{\varepsilon 1} P_k - c_{\varepsilon 2}\rho\varepsilon\right) \quad \text{where} \quad P_\varepsilon = -c_{\varepsilon 1}\frac{\varepsilon}{k}\left(\bar{U}_{i,j} + \bar{U}_{j,i}\right)\bar{U}_{i,j}$$

Turbulent length scale: $\ell = \dfrac{k^{3/2}}{\varepsilon}$ Turbulent eddy viscosity: $\nu_t = c_\mu k^{1/2}\ell = c_\mu \dfrac{k^2}{\varepsilon}$

- Robust industry standard model
- For weakly-turbulent regions k and ε tend to zero (require sublayer damping functions)
- Insensitive to inflow conditions
- Typical applications: ground transportation

Two-Equations Models

Standard $k-\omega$ model:

Two transport equations are solved for determining the turbulent kinetic energy k and the specific dissipation rate $\omega \propto \varepsilon/k$. Even though exact equations can be derived, too many unknown coefficients would make the model unpractical for computational reasons, hence the exact production and dissipation terms are simplified based on scaling considerations and on the Boussinesq hypothesis. The remaining constants of the simplified model are assumed universal.

$$(\rho \bar{U}_j k)_{,j} = \left[\left(\mu + \frac{\mu_t}{\sigma_k^\omega}\right) k_{,j}\right]_{,j} + P_k - \beta^* \omega k \quad \text{where} \quad \frac{1}{2}\overline{\rho u_j u_i u_i} = -\frac{\mu_t}{\sigma_k} k_{,j}$$

$$(\rho \bar{U}_j \omega)_{,j} = \left[\left(\mu + \frac{\mu_t}{\sigma_\omega}\right) \omega_{,j}\right]_{,j} + \frac{\omega}{k}\left(c_{\omega 1} P_k - c_{\omega 2} \rho k \omega\right) \quad \text{where} \quad \mu_t = \rho \frac{k}{\omega}, \; \varepsilon = \beta^* \omega k$$

Turbulent length scale: $\ell = \frac{k^{3/2}}{\varepsilon}$ Turbulent eddy viscosity: $\nu_t = c_\mu k^{1/2} \ell = c_\mu \frac{k^2}{\varepsilon}$

- Performs well for swirling flows
- Does not require sublayer damping functions (can be used as a low-Re number model)
- Performs well for adverse pressure gradients and wall-bounded flows
- Typical applications: aerospace, Formula 1 and turbomachinery

Two-Equations Models

SST $k-\omega$ model:

Combines the standard k–ω model, employed near walls , with the standard k–ε model away from walls by using blending functions. Turbulent viscosity is implicitly limited.

- Performs well for swirling flows
- Does not require sublayer damping functions (can be used as a low-Re number model)
- Performs well for adverse pressure gradients and wall-bounded flows
- Less sensitive to freestream turbulence than standard k–ω model
- Typical applications: aerospace, Formula 1 and turbomachinery

Realizable $k-\varepsilon$ model:

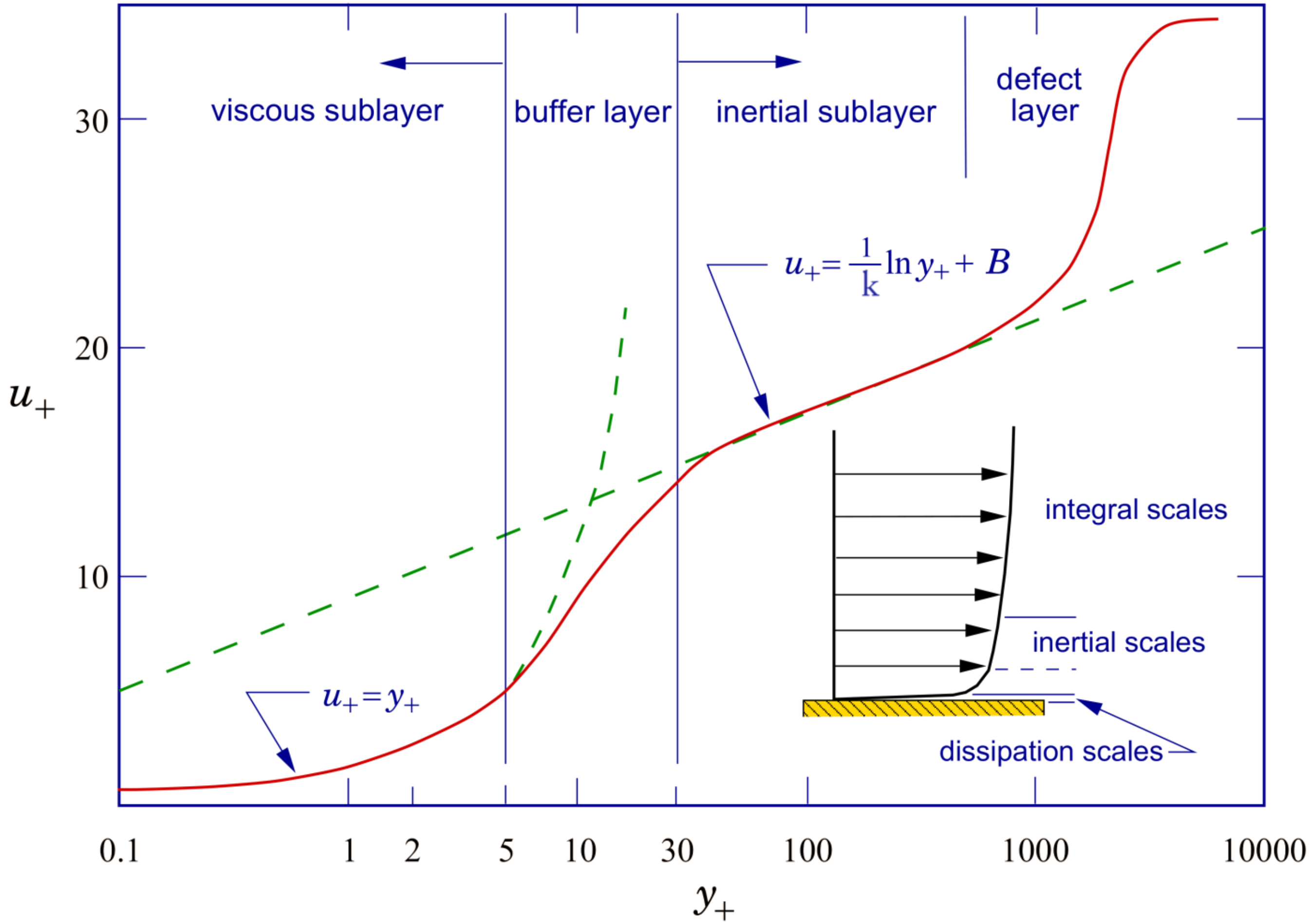
Modified version of the standard k– ε model aimed at enforcing physical constraints to the mathematical model, e.g. it ensures that normal stresses are positive.

- More physical and accurate than the standard k–ε model
- Performs better than the standard k–ε model for separated, swirling and rotating flows
- Performs better than the standard k–ε model for flows with large streamline curvature
- Typical applications: aerodynamics and ground transport

Two-Equations Models

Law of the wall:

Includes theoretical scaling arguments validated by experimental evidence. They are valid near solid walls and take advantage of separation of scales in fully turbulent flows.



$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$y_+ \equiv \frac{y u_\tau}{\nu}$$

$$u_+ \equiv \overline{u}/u_\tau$$

Two-Equations Models

Low-Reynolds-number models:

Variation of the standard turbulence models to correct for the approximations that assume high-Reynolds-number approximations of the energy budget. The low-Reynolds number models study the limit for $y \rightarrow 0$ and embed it in the standard models.

All-Reynolds-number models:

Variation of the standard turbulence models to correct for the approximations that assume high-Reynolds-number approximations of the energy budget. They match the low-Reynolds number model near the walls and the high-Reynolds-number models away from the wall.

Two-Layer k–ε model:

Variation of the standard turbulence models that use a one-equation model near the wall (they solve only for k) and match to the standard high-Reynolds model far from the wall.

Every turbulence model has a requirement in terms of the grid size compared to the y^+

Beyond Boussinesq Hypothesis

Reynolds Stress models:

In Reynolds-stress models, model transport equations are solved for the individual Reynolds stresses R_{ij} and for the dissipation rate ε (or the specific dissipation rate ω). They involve a significant computations overhead if compared to algebraic, one- and two-equations models because RSM solve for seven modeling equations (three Reynolds shear stresses, three Reynolds normal stresses, and one for dissipation), which equations must be solved in 3D.

- More physical and accurate than simpler turbulence models
- It is likely to require a higher number of iterations required to a converged solution
- Accounts for streamline curvature, swirl, rotation, and rapid changes in strain rate
- Typical applications: cyclone flows, swirling flows in combustors, rotating flow passages, and the stress-induced secondary flows in ducts

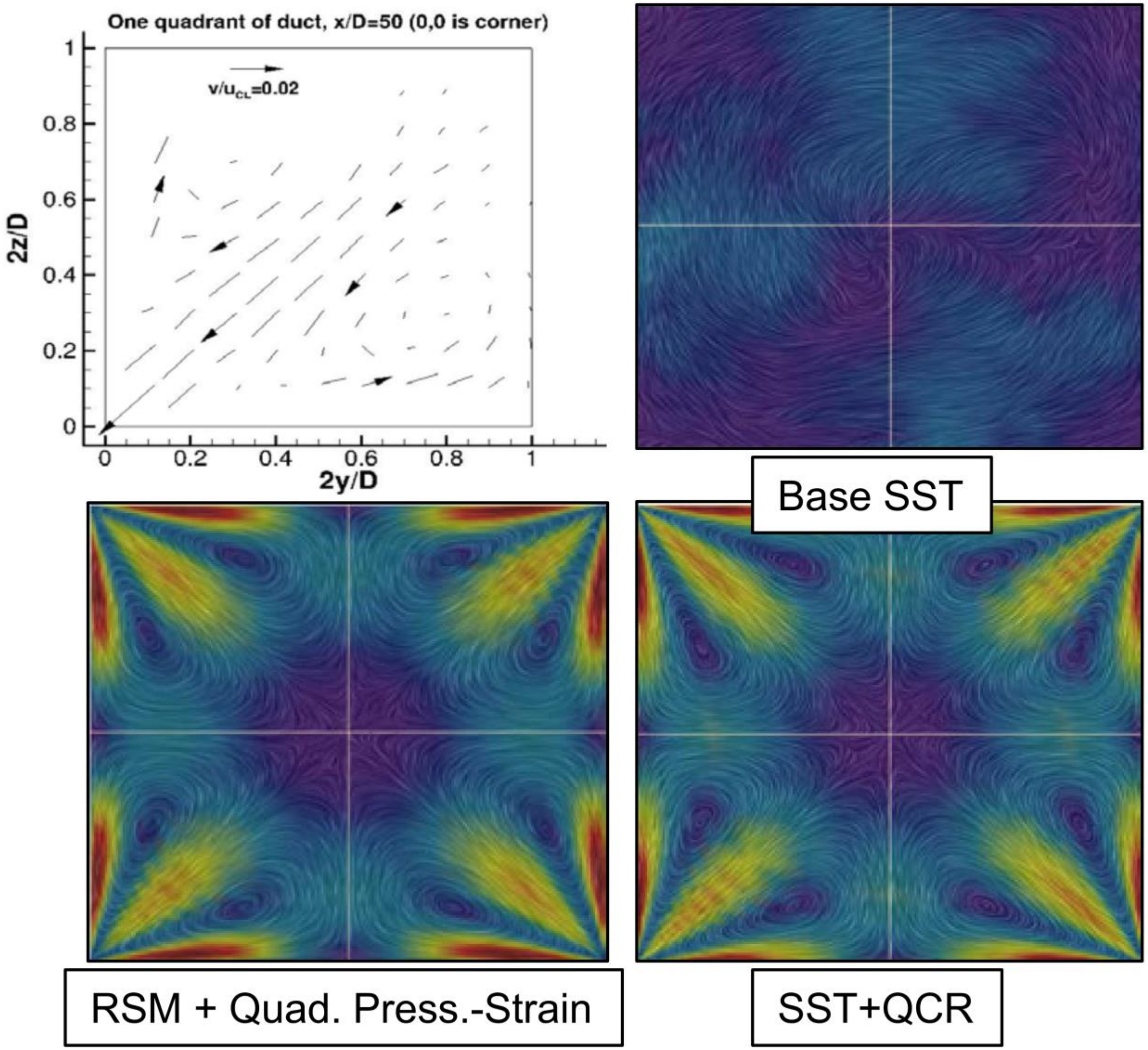
Beyond Boussinesq Hypothesis

Non-linear constitutive relations:

> The Boussinesq hypothesis assumes a linear relationship between the Reynolds-stress and the strain rate tensors. Non-linear constitutive relations instead use a higher order expansion to set up the strain-stress constitutive relation. These models do not require additional transport equations as anisotropy effects are accounted for via an algebraic formulation. At low strain rates these models recover standard model as their linear order is the turbulent eddy viscosity constitutive relation.

- More physical and accurate than simpler turbulence models
- Compatible with standard k–ε model and SST $k - \omega$ model
- Accounts for anisotropic turbulence, streamline curvature, swirl, rotation
- Well suited for boundary layer flows and when secondary flows are important
- Typical applications: strong swirling flows, aerodynamics and secondary flows in ducts

Beyond Boussinesq Hypothesis



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